A discrete approach to monogenic analysis through Radon transform

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Abstract Monogenic analysis is gaining interest in the image processing community as a true signal processing tool for 2D signals. Unfortunately, it is only defined in the continuous case. We address this issue by proposing an innovative scheme that uses a discrete Radon transform based on discrete geometry. Radon domain signal processing and monogenic analysis is studied and performance is shown to be equivalent to the usual FFT-based algorithms. The advantage is that extensions to filterbanks and to higher dimensions are facilitated, thanks to the perfect invertibility and computational simplicity of the used Radon algorithm.

1 Introduction

The monogenic framework provides efficient tensor-like local analysis of images while being linked to signal processing through a phase concept. Its building block is the Riesz transform, which performs a pure $\frac{\pi}{2}$-phase-shifting in the maximum local variation spatial direction of a 2D signal [5].

However, the definition of this promising tool involves continuous signals only. In practice, the Riesz transform is computed in the FFT domain - which corresponds to truncate and periodize its Fourier spectrum - so the monogenic analysis is in fact approximate. We believe that such a signal processing tool must be studied from a discrete viewpoint. Surprisingly we have not found any contribution in the literature about it. The work presented in this paper is based on two facts:

- There is a fundamental link between the monogenic framework and the Radon transform [9];
- 'True' discrete counterparts of Radon transforms have already been defined [2].

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More precisely, the monogenic concept is basically made of a Radon transform joint to a 1D phase analysis, so we can say that the Radon transform is responsible for *isotropy*. This crucial point has always been a deep issue in the discrete world, while at the core of monogenic analysis.

In [2] a discrete Radon transform with exact reconstruction is designed with the help of discrete analytical geometry. The present paper studies the use of this well established discrete representation to perform monogenic analysis. We expect the analysis to be numerically stable and geometrically coherent thanks to the use of discrete geometry. This work also paves the way to multiscale monogenic analysis that may be improved or simplified in the Radon domain.

The paper will first recall the monogenic analysis concept, and then the Radon transform. After that the proposed discrete Radon based monogenic analysis will be defined.

2 Monogenic analysis

Monogenic analysis consists of associating to a signal s the two components of its Riesz transform $\mathscr{R}s$. This theoretically allows extracting meaningful amplitude, phase and orientation by a conversion of the 3 terms to polar coordinates [5].

Given a 2D real (scalar) signal s, consider its Riesz transform:

$$\{\mathscr{R}s\}(x) = s_R(x) = s_{R_1} + js_{R_2} \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{\omega_2 - j\omega_1}{\|\omega\|}\hat{s}(\omega) \tag{1}$$

Note that this is the $\mathbb{C}$ embedding of the Riesz transform according to [11]. The monogenic signal s_M associated to s is vector-valued and reads:

$$s_M = [s \ \ s_{R_1} \ \ s_{R_2}] = A[\cos\varphi \ \ \sin\varphi\cos\theta \ \ \sin\varphi\sin\theta] \tag{2}$$

where $\theta = \arg\{s_R\} \in [-\pi;\pi[$ is the local Riesz orientation along which a 1D Hilbert analysis is intrinsically done. This 1D analysis can be written:

$$s_A(x) = s(x) + j|s_R(x)| = A(x)e^{j\varphi(x)} \tag{3}$$

Amplitude of the monogenic signal conveys a local presence of some geometrical elements. The θ-phase gives the corresponding local orientation (equal to a gradient direction *i.e.* direction of maximum variation in the signal s [9]). The φ-phase results from an intrinsic Hilbert analysis along orientation θ. So the signal model here is an A-strong structure that is oriented along θ and looking like rather an *edge* ($\varphi \approx \pm\pi/2$) or a *line* ($\varphi \approx 0$ or π). Note that the good discrimination of lines and edges by local phase analysis was discussed in [6, 10].

3 The Radon domain

The Radon domain represents 2D functions by a set of 1D projections at several orientations. It forms a fundamental link with 1D and 2D Fourier transforms and so handles well isotropic filtering.

3.1 Definition

Given a 2D function $s(x,y)$, its projection into the Radon domain along direction θ is defined by:

$$s_\theta(t) = \int_{\mathbb{R}} s(\ \tau \sin\theta + t\cos\theta\ ,\ -\tau\cos\theta + t\sin\theta\ ) d\tau \tag{4}$$

This representation is used for example in tomographic reconstruction, where the data is formed by spatial projections along several directions.

3.2 Properties

Note the easily verifiable following property:

$$s_\theta(-t) = s_{\theta+\pi}(t) \tag{5}$$

which ensures circular coherency of the representation.

The so-called *Fourier Slice Theorem* defines the link with the Fourier transform:

$$\hat{s_\theta}(f) = \hat{s}(f\cos\theta, f\sin\theta) \tag{6}$$

where $\hat{s_\theta}$ is the 1D Fourier transform of s_θ for a fixed θ and $\hat{s}$ is the 2D Fourier transform of s. In the Fourier domain, property (5) becomes:

$$\hat{s_\theta}(-f) = \hat{s_{\theta+\pi}}(f) \tag{7}$$

which ensures coherency of the 2D Fourier transform $\hat{s}$. This is why in practice Radon projections are restricted to $\theta \in [0;\pi[$.

The Fourier Slice theorem is helpful to derive interesting properties about Radon domain filtering.

3.3 Radon domain filtering

Given two 2D signals $s(x,y)$, $h(x,y)$ and their respective Radon transforms $s_\theta(t)$, $h_\theta(t)$, convolution in both domains is linked by:

$$(s_\theta * h_\theta)(t) = (s**h)_\theta(t) \tag{8}$$

where $*$ denote 1D convolution and $**$ denotes 2D convolution. $(s**h)_\theta(t)$ is then the Radon transform of $(s**h)(x,y)$.

If the filter h is isotropic $h(x,y) = h_1(\sqrt{x^2+y^2})$ then its Radon transform does not depend on θ and is symmetric w.r.t. t: $h_\theta(t) = h_1'(|t|)$. This means that a 1D filtering in the Radon domain with a symmetric filter is equivalent to a 2D isotropic filtering in the space domain.

Another particular case is this of the Riesz transform. It is shown in [1] that the Riesz transform is equivalent to an independent Hilbert transform (1D) on each Radon projection, combined with a sin-like weighting depending on θ:

$$\{\mathscr{R}s\}_\theta(t) = \{\mathscr{H}s_\theta\}(t)e^{j\theta} \tag{9}$$

where $\mathscr{H}s_\theta$ is the Hilbert transform of s_θ defined by:

$$\{\mathscr{H}s_\theta\}(t) = \left(s_\theta(\cdot) * \frac{1}{(\pi\cdot)}\right)(t) \tag{10}$$

The Hilbert transform is well known and already integrated for example in some *analytic* wavelet transforms [3, 8]. Performing some monogenic analysis (based on the Riesz transform) then reduces to a more classical problem in the Radon domain. We now present the algorithm implementing the discrete Radon transform.

4 Discrete Radon based Riesz transform

The work in [2] consists in defining a true discrete decomposition by using the discrete geometry. Based on the Fourier slice Theorem, discrete lines are defined in the 2D Fourier domain before performing an inverse 1D Fourier transform to each extracted line. Lines are placed in the Fourier domain according to Figure 1. An *arithmetical* thickness parameter can be used to control both redundancy and straight line connectivity. In all cases, perfect reconstruction is guaranteed and the algorithm is as simple as fast.

For our purpose, we need to compute the values of θ for each projection (in order to get the complex exponential part of equation (9)). As shown Figure 2, θ does not evolve linearly, because of the Cartesian layout of the data. Its value is computed geometrically from the scheme of Figure 1, by taking proper arc-tangents of coordinate ratios.

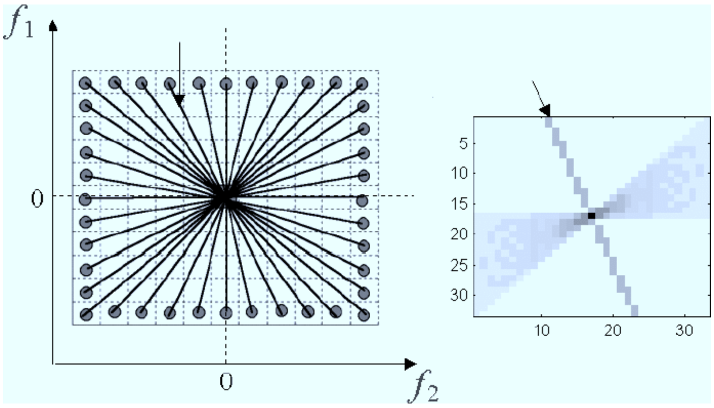

Fig. 1 Line extraction from the Fourier transform of a 2D signal.

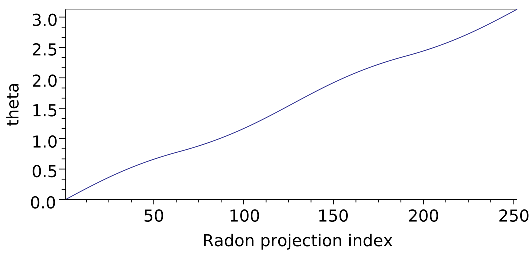

Fig. 2 Angles θ for each discrete Radon projection. The curve would be linear if we would be working on a polar grid. Here we have a usual Cartesian grid of size 127×127.

We will see that in the case of a monogenic analysis, this difference with the continuous world is not a problem and does not prevent from performing an accurate algorithm.

Computing of a discrete Radon based monogenic analysis can now be done as follows:

- Process Radon transform of s;
- Apply 1D bandpass filtering to select some scale (this is equivalent to an isotropic band-pass 2D filtering)[1];
- Process Hilbert transform of every projection s_θ;
- Multiply every projection by $e^{j\theta}$ (by using the computation of θ explained above);
- Process the inverse Radon transform on real and imaginary parts separately (we get s_{r_1} and s_{r_2});
- Convert s, s_{r_1} and s_{r_2} to spherical coordinates as in equation (2).

An example of such decomposition is given on Figure 3. The observed data is

[1] Band-pass filtering is natural in monogenic analysis that is usually presented either in a scale-space formalism or in a wavelet transform.

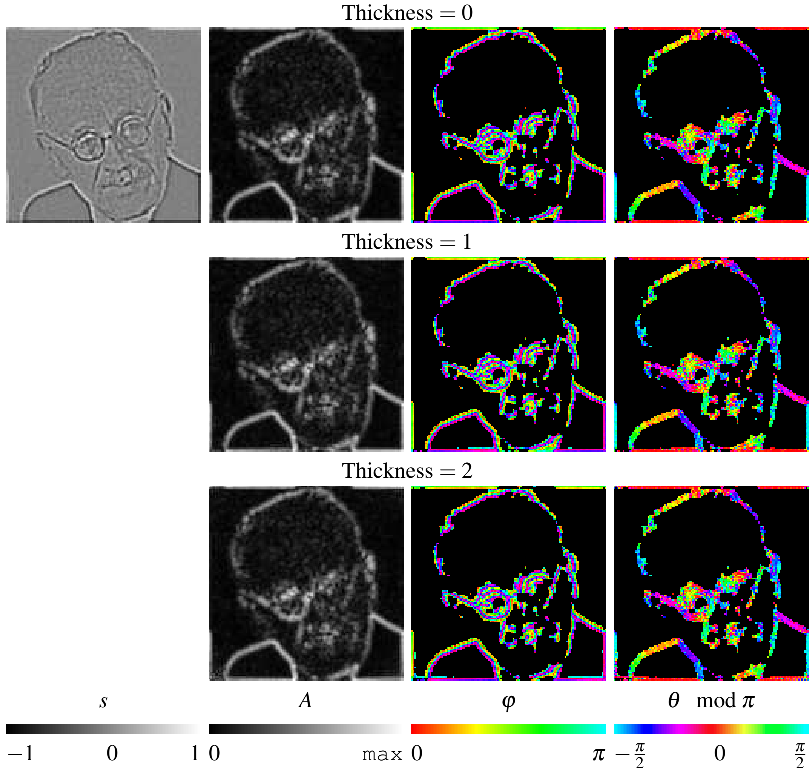

Fig. 3 Radon-based monogenic signal on a bandpass version of ‘radon’ image. Isotropic bandpass filtering is straightforwardly done in the Radon domain. Result is very analogous to FFT-based method. Black pixels at φ and θ mean insignificant phase values due to low amplitude.

very usual: amplitude A conveys local presence of elements in s, phase φ indicates the local behavior of these elements (lines are rather cos-like while edges are rather sin-like), and orientation gives the direction of maximum variation *i.e.* main orientation of the local element.

Results are given for the 3 different possible values of *arithmetical* thickness ($\in \{0,1,2\}$). Excepted some mild edge effects for 1 and 2, this parameter does not affect the quality of the Riesz transform. Differences between Radon-based and FFT-based analyses are of order 10^{-4} for amplitude and 10^{-2} for phase/orientation, with greater errors where amplitude is low due to numerical instability. In fact, the thickness parameter controls the shape and length of projections but does not change values of θ that are only dependent on the image dimensions (see Figure 1). Note that the whole scheme is fast thanks to the simplicity of the chosen Radon algorithm.

5 Discussion

This work shows that a discrete monogenic analysis can be performed in the Radon domain by using existing Radon transform algorithms. The improvement over the classical FFT-based method is not clear at this stage, but may become significant in more developed processes like wavelet transforms or extension to 3D (A 3D version of the discrete Radon transform has been defined in [7]). The existing discrete Radon transform is a good tool having exact reconstruction and computational simplicity thanks to discrete geometry. Actually, the design of monogenic filterbanks still suffers from this lack of theoretically defined discrete Riesz transform.

A solution could be to apply an analytic 1D wavelet transform in the Radon domain. However, the inverse Radon transform of a sub-sampled version of Radon projections is not defined, although necessary to get monogenic subbands in the space domain. One can then avoid inverse Radon and use Radon domain coefficients, which is equivalent to *complex ridgelets* [4]. It consists of applying the Dual-Tree complex wavelet transform on Radon slices of an image. In [4] denoising is proposed by thresholding complex ridgelet coefficients. Actually, complex ridgelet atoms have infinite support and are highly anisotropic whereas monogenic wavelet atoms would be isotropic and localized.

Monogenic curvelets have been proposed in [12] as a monogenic directional wavelet transform. Curvelet atoms are combined with their Riesz transform to improve analysis. The claimed advantage is that the phase concept is added to the high directionality of the curvelets. However, phase and directionality would have worked as well (if not better) in a non-directional transform, since the Riesz transform already performs a local orientation analysis. This is why we would rather focus on isotropic pyramid decomposition, contrary to monogenic curvelets.

In order to address the issue of a discrete use of the continuous monogenic framework, we propose an innovative scheme that uses a discrete Radon transform based on discrete geometry. The experimental equivalence with an FFT-based computation of the monogenic analysis is shown, but the prospects are more promising since the Radon domain is well handled for discrete data, as well as it extends well to higher dimensions. Exact reconstruction of the used Radon transform is also a fundamental property. The extension to monogenic filterbank could be facilitated by this method, but still needs to study discrete Radon domain sampling - which is of our prospects.

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